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Dividing an Apple into Equal Parts – An Easy Job?

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Abstract: *Theoretically seen dividing an apple (melon, potato, etc.) equally is not an easy task. For instance, with a normal knife (straight cuts) one has to hit the center so that the cut is a great circle. But there are alternatives which have strong connections to the "pizza theorem" and Cavalieri's Principle. The established theorem could be called "apple theorem".*

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Strictly speaking dividing an apple (melon, potato, etc.) into equal parts is not as easy as it may seem at first glance. Even if it is to be shared only between two people and the apple is a perfect sphere. After all, one has to hit the center so that the cutting area is a great circle. Cutting the apple into roughly equal pieces will not be a problem at all in real life. There will normally be no conflict over who gets which piece. But what if the pieces are to be completely exact? Of course, such considerations are more theoretical than practical in nature, but they may provide useful mathematical and didactical input for teaching mathematics at different levels. In fact, in mathematics important questions are not always practical, but in some cases more theoretical.

With complete analogous words a paper concerning the "pizza theorem" starts – Humenberger 2015. Now we are one dimension higher, in the three dimensional space, we have a ball (sphere) instead of a disc (circle). The core of this short paper is to establish an interesting three dimensional equivalent to the pizza theorem. We need not consider formal treatments of the phenomenon (long calculations, abstract proofs), we primarily will have to apply Cavalieri's Principle.

I came across to this generalization because my friend and former colleague B. Schuppar (Dortmund) sent me a problem from the so called "Bundeswettbewerb Mathematik" (Germany, 2008, 2nd round, translated):

Problem 3: Through an inner point of a sphere there are placed three planes which are perpendicular to each other. These planes divide the surface area of the sphere into eight "curved triangles". The triangles are colored alternately black and white so that the surface looks like chess board.
Prove that then exactly the half of the surface area of the sphere is colored black.

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This immediately reminded me of the “pizza theorem”. Can we use the cutter described in Humenberger 2015 also to divide a sphere (apple, melon, potato, . . .) equally?

Fig. 1a: Dividing a pizza

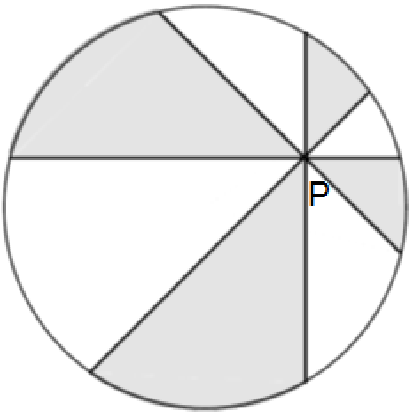


Fig. 1b: The cutter – schematically

The surprising pizza theorem states that for all positions of *P* (within the disc, Fig. 1b) the gray areas together are exactly as big as the white areas together, both are half of the disc (also the length sums of the gray and white pizza boundaries are equal, both are half of the circle perimeter).

When we imagine that we divide a spherical apple with such a cutter (the axis of the cutter needs not to pass through the center, it can hit the apple also somewhere “decentralized”) we get 8 wedges. One can imagine further to color every second wedge gray and the others white, then – projected onto the horizontal plane – one gets the analogous picture as in the pizza theorem (Fig. 1b).

A three dimensional picture would look like Fig. 2.

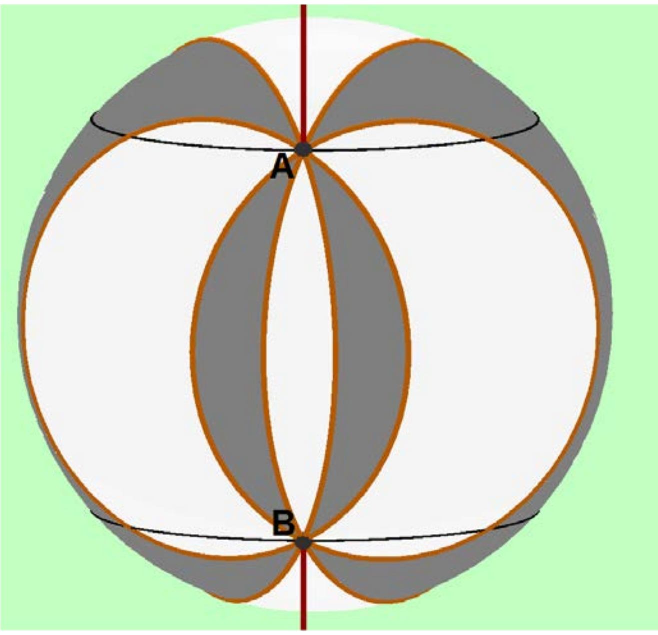


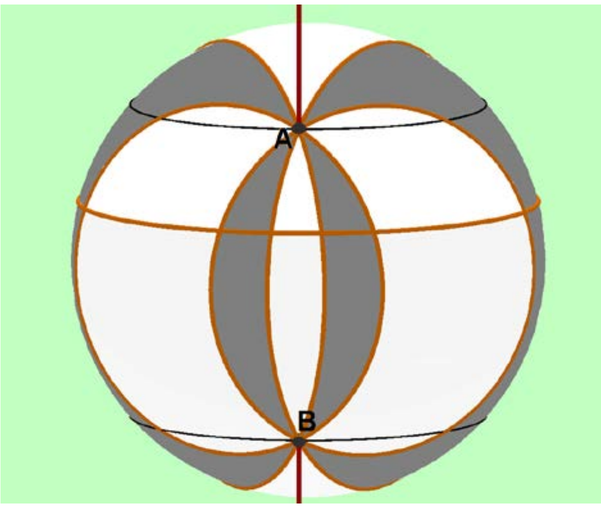
Fig. 2: Dividing an apple in wedges, each of them having 45°

A and *B* mark those points in which the "center" of the cutter, i.e. the axis of the cutter, "hits" the sphere and "leaves" it respectively. Viewing from "above" in the plane projection there would be the point *P* (Fig. 1b). Now it is nearby to ask for a three dimensional analogon of the pizza theorem: Are the volume sums (or the surface area sums) of gray and white equal? Sometimes? Always? Never? It is clear from the very beginning: If one blade of the cutter hits the center of the sphere then the equality of gray and white is given by symmetry. But if the axis of the cutter hits the sphere near the "equator" (i.e. *A* and *B* in Fig. 2 are very close one above the other) it is evident that the white part of the sphere surface area which is in Fig. 2 nearly invisible in the rear part becomes pretty big and that white therefore will make more than the half surface area of the sphere.

That is maybe disappointing at the first glance. Because after all at each horizontal cross-section between the two "circles of latitude" passing through *A* and *B* one can observe a constellation like in the plane pizza theorem, and from the plane pizza theorem we know the area equality between gray and white. Following *Cavalieri's Principle*[2] we can conclude immediately that the volumes of gray and white are equal between these two "circles of latitude". Using an analogous argument as in the plane pizza theorem (see appendix) one can conclude: In the addressed zone of the sphere we also have *equality* of *surface areas* between gray and white. The reason for the missing equality on the whole must lie somehow in the "polar regions", i.e. in the "north" of the circle of latitude through *A* and in the south of the circle of latitude through *B*. These two "polar regions" are perfectly symmetric, that means all the gray parts in the north of *A* have their gray counterpart in the south of *B* (and the same applies for the white areas). Therefore if there is a balance between gray and white on the whole, then each polar region must be balanced, but: are they?

[2] This states: Suppose two solids are included between two parallel planes. If every plane parallel to these two planes intersects both solids in cross-sections of equal area, then the two solids have equal volumes.

One needs not to think a lot about this problem whether there is balance in the polar regions (except in the trivial cases in which each "polar region" is built up symmetrically with respect to gray and white – one blade of the cutter meets the center of the sphere) because with a further (horizontal) cut in an arbitrary "height" somewhere between the two circles of latitude through *A* and *B* – hereby each wedge is cut into two parts horizontally, see Fig. 3a – one can enforce this balance: By changing all the colors in the upper (one could also take the lower) part one "polar region" is clearly "reversed with respect to colors", so that there is perfect balance between the two "polar regions" (and the balance in the region between the two circles of latitude through *A* and *B* still exists), see Fig. 3b.



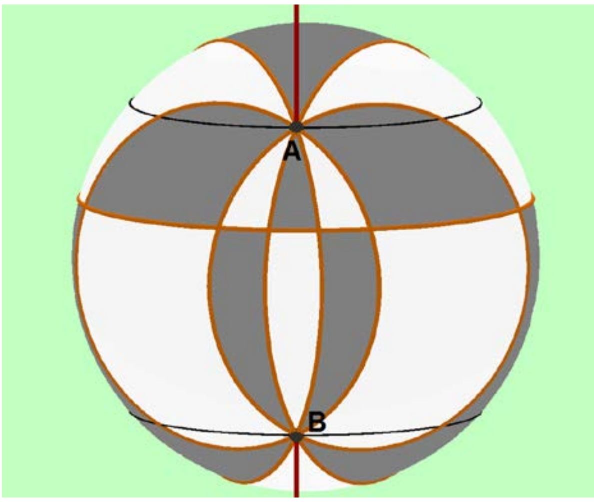


Fig. 3a: additional horizontal plane

Fig. 3b: After changing the colors in the upper part

The following technique would yield a perfectly fair partition of an apple (melon, potato, etc.; without necessarily hitting the center): First divide the apple with a nearly arbitrary horizontal cut in two parts, this cut can but needs not hit the center. If then the pizza cutter (45° angles) is used vertically so that the axis (center) of the cutter hits the upper part of the fruit, then $2 \times 8 = 16$ parts (divided wedges) are generated, which can be used for a perfectly fair partition between two persons: One person takes all the "gray" parts the other one all the white ones. Both persons then have also an equal amount of apple skin (surface area; the apple core is not divided fairly in the general case; but this problem does not arise in the case of melons or potatoes).

Here it is important that the first cut is *horizontal* and that the cutter is on the one hand pressed down *vertically* onto the fruit and on the other hand has exact 45° angles. It is obvious that these requirements are not met trivially but one needs not to hit the center. But actually theoretical aspects are more important here than practical ones.

Just like in the pizza theorem also here an interesting phenomenon – one dimension higher – arises: If one wants to have a fair partition of only the *surface area* then a cutter can be taken that has only two (orthogonal) blades (in the cited problem above from the "Bundeswettbewerb Mathematik" there were three orthogonal planes, two of them representing the blades in the context of the cutter, the third one the above mentioned horizontal plane). If one wants to have also a fair partition of the volume one needs at least four blades on the cutter (45° angles), i.e. each “quadrant” has to be bisected (with respect to angles). This is because in the plane pizza theorem one needs four blades (cf. Fig. 1b) in order to have constant area sums of gray and white respectively. Constant sums of arc lengths in the plane pizza theorem we can have already with two blades (see also the remark in Humenberger 2015, p. 394).

The pizza theorem holds also for other cutters, i.e. each quadrant is divided in 3, 4, 5, . . . parts with equal angles (instead of 8 parts we then have altogether $12, 16, 20, \ldots$ parts of the pizza, cf. Humenberger 2015, p. 399ff) – see Fig. 4 with three parts per quadrant. Also such cutters yield a fair partition of an apple between two persons .

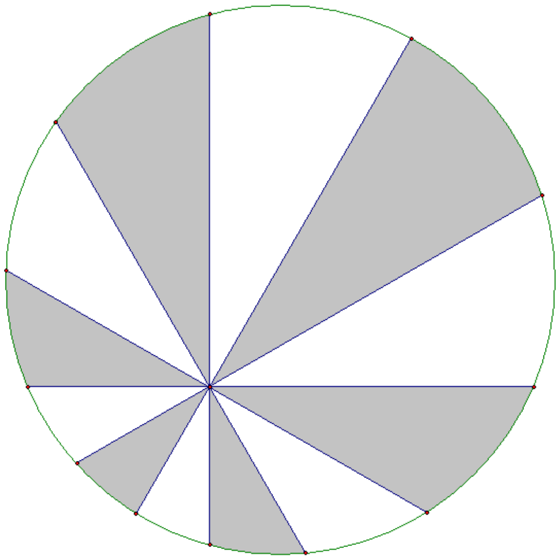


Fig. 4: Fair division of a pizza with 3 parts per quadrant

This aspect can be transferred in a natural way to the fair division of an apple, and this yields the

Apple Theorem: An apple (melon, potato, mathematically seen as a sphere) is cut first horizontally (in Fig. 5 this is shown in vertical and horizontal projection). Then the apple is divided using an equally angled cutter (each quadrant has equally many parts – at least two – with equal angles), which is pressed vertically from above onto the apple. The center (axis) of the cutter hits the “upper part” in an arbitrary point. If then the surface area of the fruit is colored like a chess board the following holds: The volumes of all gray parts together are the same as the volumes of all white parts together. An analogous phenomenon holds for the surface areas instead of the volumes. Therefore, one has a perfectly fair division between 2 persons.

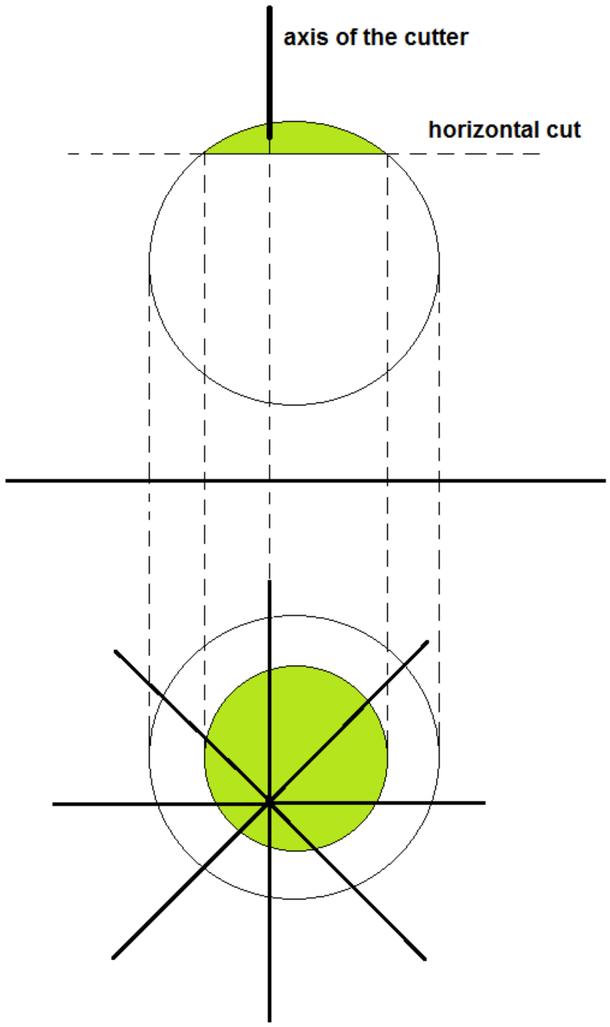


Fig. 5: Apple Theorem

On the one hand this phenomenon can be seen as a beautiful application of *Cavalieri's Principle*. The proof of the pizza theorem is not quite easy, but this generalization to the three dimensional space (*volumes*, partition of an apple) needs neither further calculations nor further abstract or formal proofs, Cavalieri's Principle suffices and the idea of a further horizontal cut in order to bring the polar regions to opposite colors. On the other hand when dealing with the fair partition of the *surface area* again tedious calculations are not necessary if one thinks of the surface area as a "infinitely thin" hollow sphere (the equality of the volumes directly implies the equality of the surface areas; for details see the appendix). Altogether in dealing with both phenomena (volumes, surface areas) one comes to beautiful and perhaps surprising results without further calculations (if the pizza theorem is known) by using important principles and concepts.

Unfortunately one cannot realize a fair partition between more than two persons in that way because there are only two polar regions. In the zone between the two mentioned circles of latitude (through *A* and *B*) one could easily realize a fair partition (using a cutter that divides each quadrant in more than two equally angled parts[3]) but the simple trick of changing colors in one polar region unfortunately does not work anymore in case of more than two colors (persons).

Using GeoGebra (3D) one can easily intersect spheres and planes, so that the corresponding circles (in the picture they are ellipses) and primarily the corresponding parts of the sphere's surface area ("curved triangles") can be made visible.

Unfortunately one cannot calculate the volumes and the surface areas of the pieces easily, also when using GeoGebra this is not easy. Another thing that is not easy in GeoGebra is to have different colors for different pieces[4], so that pictures as in Fig. 3 cannot be produced within GeoGebra. That means one cannot expect that students discover the theorem by themselves just doing experiments with Dynamics Geometry Software (as a means for measuring). But a problem of the following kind could be solved by students (e.g. in a lecture or seminar concerning geometry or problem solving): How can we use the *pizza theorem* to prove the *apple theorem*? Here one could give a hint – depending on the level of performance of the students – mentioning Cavalieri's Principle, the idea of changing colors in one polar region would have to come from the students. For "real experiments" with such dissections one can use real apples (melons, potatoes) and a sharp knife, in order to do "hands on geometry", and not only geometry with objects that are merely in our mind.

An investigation to this topic brought up that also other mathematicians promoted this idea. For instance we found[5] in George Berzsenyi's text (1994) that this phenomenon was discovered by Michael Nathanson[6] as a freshman student at Brown university, he established the "Calzone Theorem" (this notation probably is due to the fact that in America a "Calzone" often is shaped like a *sphere* – filled with some sorts of cheese): "Choose any point P inside or on the boundary of a sphere (calzone), any line through this point, and four planes through this line making eight equal 45° angles at P. Then these planes, together with the plane perpendicular through this line, divide the calzone into 16 pieces, which can be colored alternately black and white, so that the total volume of the black pieces will be equal to the total volume of the white pieces. The proof can be obtained by using Cavalieri's Principle."

References:

Humenberger, H. (2015): Dividing a pizza into equal parts – an easy job? The Mathematics Enthusiast, vol. 12 [issues 1, 2 & 3, June 2015], pp. 389–403.

Berzsenyi, G. (1994): The Pizza Theorem – Part II. In: Quantum, Vol. 4, Nr. 4 (April/March 1994), p. 29. (http://static.nsta.org/pdfs/QuantumV4N4.pdf)

[3] In the plane pizza theorem one can have a partition between more than two persons (cf. Humenberger 2015, p. 396).

[4] Coloring the different parts differently is possible with GeoGebra when using more advanced "tricks". For the figures 2 and 3 we used another program for coloring.

[5] Other references we did not find.

[6] Now professor at St. Mary's College, California. In a private email he wrote to me that he never published a proof of this theorem, but he gave several talks to the topics "pizza theorem" and "calzone theorem" in the last decades.

Appendix:

More detailed reasoning for the phenomenon that the surface area ("apple skin") is also equally divided

Due to the process of changing colors in one of the polar regions it is guaranteed that within the polar regions we have area balance between gray and white. We have to show that also in the region between the two circles of latitude A and B we have the mentionesd area balance (see above). We will see that the argumentation runs analogously to the plane pizza theorem (cf. Humenberger 2015, 398f), just "one dimension higher": Instead of arc lengths and areas we have here surface areas and volumes.

Region between the circles of latitude through *A* and *B*

We have already mentioned that – with arbitrary radii – the gray volume is independent of the point's P position (axis of the cutter) and of the position (concerning rotation) of the blades, it is always half of the total volume. Hence the same holds for the difference of two such volumes, that is for the volume sum $\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3 + \Delta V_4$ of the four grey parts of the hollow sphere when increasing the radius from r to $r + \Delta r$ (Fig. 6; there only a two dimensional cross-section is shown; the surface areas A_1, A_2, A_3, A_4 appear as arc lengths).

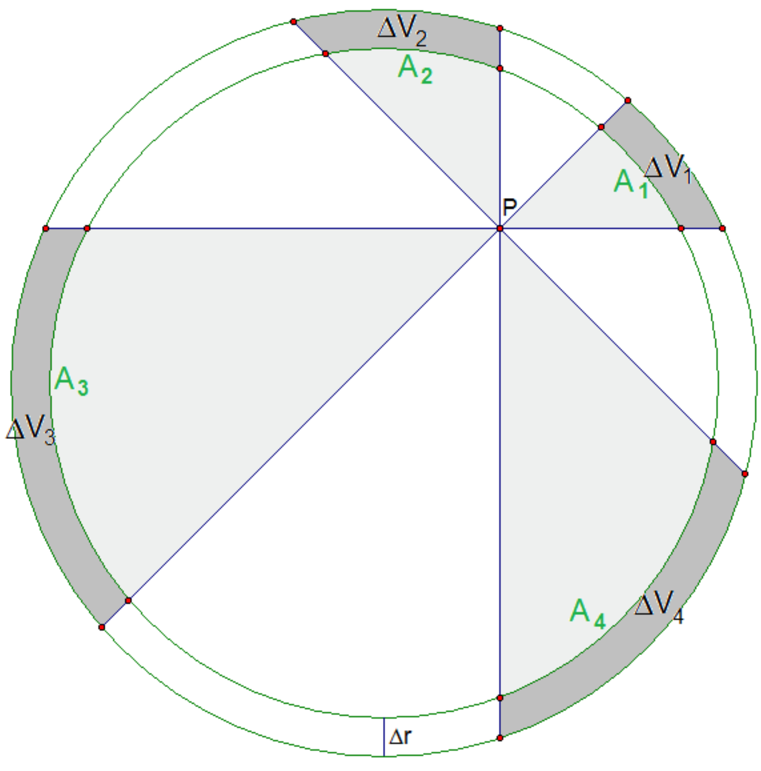


Fig. 6: Fair division of the surface area ("apple skin")

Since $\Delta V \approx (A_1 + A_2 + A_3 + A_4) \cdot \Delta r$ (the hollow sphere has everywhere the same thickness Δr, hence these parts of the hollow sphere together can approximately be thought of as a cylinder with base area $A_1 + A_2 + A_3 + A_4$ and height Δr) does not change under translation or rotation of the cutter (see above) the same holds for $\frac{\Delta V}{\Delta r}$ and in the limit also for

$\lim_{\Delta r \to 0} \frac{\Delta V}{\Delta r} = \frac{\mathrm{dV}}{\mathrm{d}r} = A_1 + A_2 + A_3 + A_4$ (sum of the four gray surface areas). That means also the sum of the four gray surface areas $A_1 + A_2 + A_3 + A_4$ ("apple skin") is independent of the point's *P* position and of the rotation position of the blades. Therefore this sum is half of the total surface area, the other half is white.

The mathematical background of this phenomenon is the fact that the derivative of the volume of a sphere (with respect to the radius) is the surface area (in the context of the pizza we had the idea of the circumference as the derivative of the area). These are important and basic concepts – geometrically and didactically: Not only confirm $\frac{\mathrm{d}V}{\mathrm{d}r} =$ surface area by using the formulas ($V = \frac{4\pi}{3} \cdot r^3$, surface area $= 4\pi \cdot r^2$ and in case of the circle $A = \pi \cdot r^2$, perimeter $= 2\pi \cdot r$) and formal differentiation rules but to be able to explain these relations with regards to contents (analogous in the case of the circle). Above we used this phenomenon not for the whole sphere but only for a part of it (the part which lies between the two circles of latitude through *A* and *B*, see above).